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The generalized Hulthén potential

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Abstract. A generalization of the Hulthén potential is presented on the basis of an approach that uses the factorization of a general Hamiltonian by means of a specific model of operational equations with the structure $\sim\beta(r) \mp (\mathrm{d}/\mathrm{d}r)$. To achieve this goal, the treatment of the $V_{\mathrm{H_s}}(r)$ standard Hulthén potential for bound s states is carried out by proposing a particular $\beta_{\mathrm{p}}(r)$ ansatz to identify $V_{\mathrm{H_s}}(r)$ by means of a particular Riccati-type relationship. Once the identification has been achieved, the generalized Hulthén potential is obtained straightforwardly with the aid of a general Riccati formula. As expected, the Hamiltonian of the generalized Hulthén potential is isospectral when compared with the corresponding standard Hamiltonian. Moreover, according to the Darboux transform there exists a modified Hulthén potential which is also isospectral. We show that the latter is just a particular case of the generalized Hulthén interaction model.

Key words: Isospectral Hamiltonians – Darboux's transformation – Molecular potentials – Schroedinger equation

1 Introduction

The Hulthén potential [1] is a useful interaction model that has been used extensively in different areas of physics, including nuclear [2] and atomic physics [3], due to the fact that it yields closed analytic solutions for the s waves [4]. In the solution of the case of $\ell \neq 0$, different approaches have appeared, namely the dynamic-group [5] and the Padé approximation methods [6]. Moreover, the algebraic treatment of the Hulthén potential has been given by de Lange [7] as well as the calculation on certain expectation values using the hypervirial theorem [8]. Also, as a consequence of Mielnik's discovery of a family of harmonic oscillator isospectral potentials [9], other standard potentials have been used in the literature in order to get the corresponding isospectral Hamiltonians [10–12]. In spite of these efforts, as far as we know, the isospectral Hulthén potential has not been found yet. However, recently Morales and coworkers [13, 14] proposed a useful general method to obtain the generalized or isospectral potential that corresponds to a known standard potential. The generalization of the Hulthén potential is given in this work by using the aforementioned procedure. In order to do that, it is important to keep in mind that when considering bound s states, the suggested approach is based on the existence of an algorithm that uses two specific formulae: the first one is given by the Riccati relationship,

$$V_{\mathrm{p}}(r) = \frac{\hbar^2}{2m}\left[\beta_{\mathrm{p}}^2(r) + \beta_{\mathrm{p}}'(r)\right] - C_{\mathrm{p}} \ , \qquad (1)$$

where $\beta_{\mathrm{p}}(r)$ is an ansatz needed to identify the particular potential, $V_{\mathrm{p}}(r)$, under study, $'$ indicates a total derivative and C_{p} is a constant parameter used to clean $V_{\mathrm{p}}(r)$, and the second one is

$$V_{\mathrm{g}}(r) = V_{\mathrm{p}}(r) - \frac{\hbar^2}{m}\frac{\mathrm{d}}{\mathrm{d}r}\left(\beta_{\mathrm{p}}(r) + \frac{b}{\rho(r)}\right) \ , \qquad (2)$$

where

$$\rho(r) = \exp\left[2\int \beta_{\mathrm{p}}(r)\mathrm{d}r\right]\left\{\gamma + b\int \exp\left[-2\int \beta_{\mathrm{p}}(r)\mathrm{d}r\right]\mathrm{d}r\right\} \qquad (3)$$

and $V_{\mathrm{g}}(r)$ is the generalized potential to be determined.

2 The generalized Hulthén potential

With the aim of generalizing the radial Hulthén potential for bound s states, it is important to point out that according to the proposal displayed in Sect. 1, $\beta_{\mathrm{p}}(r)$ is equivalent to the superpotential of Dutt et al. [15]. As a

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consequence, it becomes necessary to use in Eq. (1) the following ansatz

$$\beta_p(r) = \frac{A}{e^{Ar}-1} - k \ , \tag{4}$$

where A and k are arbitrary constants, in order to get

$$V_p(r) = -\frac{\hbar^2}{2m}\left(\frac{A^2+2kA}{e^{Ar}-1}\right) + \frac{\hbar^2 k^2}{2m} - C_p \ . \tag{5}$$

Thus, by putting $U_0 = \frac{\hbar^2}{2m}\left(A^2+2kA\right)$ in the above relationship one obtains

$$V_p(r) = -\frac{U_0}{e^{Ar}-1} + \frac{\hbar^2 A^2}{8m}\left(\frac{2mU_0}{\hbar^2 A^2} - 1\right)^2 - C_p \ . \tag{6}$$

This last potential gives rise to the standard Hulthén potential [7]

$$V_{H_s}(r) = -\frac{U_0}{e^{Ar}-1} \ , \tag{7}$$

where H_s refers to the standard Hulthén potential, only on the condition that $C_p = (\hbar^2 A^2/8m)((2mU_0/\hbar^2 A^2) - 1)^2$. That is, the standard Hulthén Hamiltonian, H_{H_s} is factorized according to

$$a^- a^+ = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\left[\beta_p^2(r) + \beta_p'(r)\right] \ , \tag{8}$$

by

$$\begin{aligned} a^- a^+ &= -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - \frac{U_0}{e^{Ar}-1} + \frac{\hbar^2 A^2}{8m}\left(\frac{2mU_0}{\hbar^2 A^2} - 1\right)^2 \\ &= H_{H_s} + C_p \ , \end{aligned} \tag{9}$$

where

$$a^\pm = \frac{\hbar}{\sqrt{2m}}\left(\beta_p(r) \mp \frac{d}{dr}\right) \ . \tag{10}$$

Consequently, in order to obtain the generalized Hulthén potential we proceed as follows. According to Eq. (2) we have

$$\begin{aligned} V_{H_g}(r) = &-\frac{U_0}{e^{Ar}-1} + \frac{\hbar^2 A^2}{m}\frac{e^{Ar}}{(e^{Ar}-1)^2} \\ &-\frac{\hbar^2}{m}\frac{d}{dr}\left(\frac{b}{\rho_H(r)}\right) \ , \end{aligned} \tag{11}$$

where, from Eqs. (3) and (4)

$$\begin{aligned} \rho_H(r) = &\exp\left[-2kr + 2\ln(1-e^{-Ar})\right] \\ &\times \left\{\gamma + b\int \exp\left[2kr - 2\ln(1-e^{-Ar})\right]dr\right\} \ . \end{aligned} \tag{12}$$

Besides, it is important to note that

$$\exp\left[\mp 2kr \pm 2\ln\left(1-e^{-Ar}\right)\right] = \left(\frac{e^{Ar}-1}{e^{(A+k)r}}\right)^{\pm 2} \ , \tag{13}$$

for which $\rho_H(r)$ is rewritten as

$$\rho_H(r) = \frac{(e^{Ar}-1)^2}{e^{A(v^2+1)r}}\left(\gamma + b\int \frac{e^{A(v^2+1)r}}{(e^{Ar}-1)^2}dr\right) \ , \tag{14}$$

with $v^2 = 2mU_0/\hbar^2 A^2$. The derivative of $\rho_H(r)$ is given by

$$\rho_H'(r) = 2\beta_p(r)\rho_H(r) + b \ , \tag{15}$$

which leads to

$$\frac{d}{dr}\left(\frac{b}{\rho_H(r)}\right) = -\frac{2b\left(\frac{A}{e^{Ar}-1} - k\right)}{\rho_H(r)} - \frac{b^2}{\rho_H(r)^2} \tag{16}$$

that should be used in Eq. (11) in order to obtain explicitly the generalized Hulthén potential.

3 Isospectrality of the generalized Hulthén Hamiltonian

In order to prove the isospectrality of the Hamiltonian associated with the generalized Hulthén potential, we can proceed in at least two ways: step by step by applying the generalized Hulthén potential Hamiltonian to the corresponding wavefunctions or proving the isospectrability of the Hamiltonian connecting to any generalized arbitrary potential. The second option is more general and is related to the fact that Eq. (2) is written as

$$V_g(r) = V_p(r) - \frac{\hbar^2}{m}\beta_p'(r) + \frac{b\hbar^2}{m}\frac{\rho'(r)}{\rho(r)^2} \tag{17}$$

in such a way that

$$H_g = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V_p(r) - \frac{\hbar^2}{m}\beta_p'(r) + \frac{b\hbar^2}{m}\frac{\rho'(r)}{\rho(r)^2} \ . \tag{18}$$

Besides, it should be noted that Eq. (2) comes from the solution of the generalized Riccati relationship

$$V_g(r) = \frac{\hbar^2}{2m}\left[\beta_g^2(r) + \beta_g'(r)\right] - C \ , \tag{19}$$

which is given by

$$\beta_g(r) = \beta_p(r) + \frac{b}{\rho(r)} \ . \tag{20}$$

As a consequence, by using the equivalent of Eq. (10), the generalized operators

$$a_g^\pm = \frac{\hbar}{\sqrt{2m}}\left(\beta_p(r) + \frac{b}{\rho(r)} \mp \frac{d}{dr}\right) \tag{21}$$

give rise to the generalized wavefunctions

$$\varphi_g = a_g^+\psi_p = \frac{\hbar}{\sqrt{2m}}\left(\beta_p(r)\psi_p + \frac{b}{\rho(r)}\psi_p - \psi_p'\right) \ , \tag{22}$$

where ψ_p are the particular or specific eigenfunctions of H_p. These generalized operators factorize both the particular H_p and the generalized H_g Hamiltonians by means of

$$a_g^- a_g^+ = H_p + C \tag{23}$$

and

$$a_g^+ a_g^- = H_g + C \ . \tag{24}$$

Thus, by applying the generalized Hamiltonian to the generalized wavefunctions one finds

$$H_g\varphi_g = \frac{\hbar}{\sqrt{2m}}\left[-\frac{\hbar^2}{2m}\left[\left(\beta_p''(r) - \frac{b\rho''(r)}{\rho(r)^2}\right)\psi_p + \left(\beta_p(r) + \frac{b}{\rho(r)}\right)\psi_p'' - \psi_p'''\right] - \frac{\hbar^2}{m}\left(\frac{b\rho'(r)^2}{\rho(r)^3} + \beta_p(r)\beta_p'(r) + \frac{b\beta_p'(r)}{\rho(r)} - \frac{b\beta_p(r)\rho'(r)}{\rho(r)^2} - \frac{b^2\rho'(r)}{\rho(r)^3}\right)\psi_p + \left(\beta_p(r) + \frac{b}{\rho(r)}\right)V_p(r)\psi_p - V_p(r)\psi_p'\right] \ . \quad (25)$$

As can be appreciated, this last relationship contains the first-, second- and third-order derivatives of the wavefunction as well as first- and second-order derivatives of the ansatz $\beta_p(r)$ and $\rho(r)$. For that, in order to simplify Eq. (25) we begin by using the Schroedinger equation

$$-\frac{\hbar^2}{2m}\psi_p'' + V_p(r)\psi_p = E_p\psi_p \quad (26)$$

in order to obtain its derivative

$$\frac{\hbar^2}{2m}\psi_p''' = V_p'(r)\psi_p + V_p(r)\psi_p' - E_p\psi_p' \ . \quad (27)$$

With this last identity, Eq. (25) is simplified as

$$H_g\varphi_g = \frac{\hbar}{\sqrt{2m}}\left[-\frac{\hbar^2}{2m}\left(\beta_p''(r) - \frac{b\rho''(r)}{\rho(r)^2}\right)\psi_p + \left(\beta_p(r) + \frac{b}{\rho(r)}\right)E_p\psi_p - E_p\psi_p' + V_p'(r)\psi_p - \frac{\hbar^2}{m}\left(\frac{b\rho'(r)^2}{\rho(r)^3} + \beta_p(r)\beta_p'(r) + \frac{b\beta_p'(r)}{\rho(r)} - \frac{b\beta_p(r)\rho'(r)}{\rho(r)^2} - \frac{b^2\rho'(r)}{\rho(r)^3}\right)\psi_p\right] \ . \quad (28)$$

Similarly, in Eq. (28) we use, from Eq. (1), the derivative of the Riccati relationship

$$V_p'(r) = \frac{\hbar^2}{2m}\left[2\beta_p(r)\beta_p'(r) + \beta_p''(r)\right] \quad (29)$$

in order to get

$$H_g\varphi_g = \frac{\hbar}{\sqrt{2m}}\left[\frac{\hbar^2}{2m}\left(\frac{b\rho''(r)}{\rho(r)^2}\right)\psi_p + \left(\beta_p(r) + \frac{b}{\rho(r)}\right)E_p\psi_p - E_p\psi_p' - \frac{b\hbar^2}{m} \times \left(\frac{\rho'(r)^2}{\rho(r)^3} + \frac{\beta_p'(r)}{\rho(r)} - \frac{\beta_p(r)\rho'(r)}{\rho(r)^2} - \frac{b\rho'(r)}{\rho(r)^3}\right)\psi_p\right] \ . \quad (30)$$

At this point, the use of Eq. (15) and its derivative

$$\rho''(r) = 2\beta_p(r)\rho'(r) + 2\beta_p'(r)\rho(r) \quad (31)$$

in Eq. (30) leads to

$$H_g\varphi_g = \frac{\hbar}{\sqrt{2m}}\left[\left(\beta_p(r) + \frac{b}{\rho(r)}\right)E_p\psi_p - E_p\psi_p'\right] \ . \quad (32)$$

Factorizing the energy in this last relationship, the term within {..} is identified straightforwardly with the product of $(\hbar/\sqrt{2m})$ and φ_g, given in Eq. (22), for which

$$H_g\varphi_g = H_g(a_g^+\psi_p) = E_p\varphi_g \ , \quad (33)$$

proving that the generalized Hamiltonian, H_g, is also isospectral independent of the potential under study.

4 The Darboux's potential for the Hulthén model

According to Korolev [16], the Darboux transform is closely related to the notion of an isospectral transform of the Schroedinger operator. This happens because the Darboux transform [17] establishes that if a solution $\psi_0(x)$ of the Schroedinger equation for a given operator H_0 is known, then this transform provides a family of operators H whose spectrum coincides with that of H_0. In the case of this work, according to the procedure of Morales and coworkers [13, 14], the factorization

$$a^+a^- = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2}{2m}\left[\beta_p^2(r) - \beta_p'(r)\right] \quad (34)$$

gives rise, in general, to the existence of a modified Hamiltonian, $H_m(r)$, given by

$$H_m(r) = a^+a^- - C_p = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_m(r) \ , \quad (35)$$

where $V_m(r)$ is the corresponding modified potential

$$V_m(r) = V_p(r) - \frac{\hbar^2}{m}\beta_p'(r) \ .$$

This modified potential is exactly the same, as the one obtained when using the Darboux transformation [17] and we named it the Darboux potential. That is, for the Hulthén model its Darboux potential is given by

$$V_{H_m}(r) = -\frac{U_0}{e^{Ar} - 1} + \frac{\hbar^2A^2}{m}\frac{e^{Ar}}{(e^{Ar} - 1)^2} \quad (36)$$

in such a way that the $H_m(r)$ Hulthén modified Hamiltonian, or Darboux Hamiltonian for the Hulthén model, is related to the $H_p(r)$ Hulthén particular Hamiltonian by means of

$$H_p(r)a^- = a^-H_m(r) \quad \text{and} \quad H_m(r)a^+ = a^+H_p(r) \ , \quad (37)$$

which leads to

$$H_m(r)a^+\psi = E_pa^+\psi \ , \quad (38)$$

indicating, as expected, the isospectrality of $H_m(r)$.

Finally, we conclude by remarking that the procedure used to generalize a standard potential is a straightforward method, which is far simpler than equivalent approaches used to find new families of isospectral known potentials [18]. Moreover, by finding the modified Hulthén potential we have implicity shown that the so-called new families of isospectral known potentials are only a particular case of modified potentials because they match with the corresponding Darboux potentials. Therefore, the generalized potentials studied here are totally different from the Darboux potentials, although these last potentials can be considered as particular cases of partner potentials.

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